CLOSED FORM FORMULAS FOR ELF INDUCTIVE COUPLING BETWEEN FINITE LENGTH WIRES WITH EARTH RETURN

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Abstract: This paper deals with the evaluation of the inductive coupling between an overhead finite length wire circuit and a buried wire circuit, both with earth return, in the range of Extremely Low Frequencies (ELF). In particular, the attention is focused on the key parameter for this kind of problems i.e. the per unit length (p.u.l.) mutual impedance between the two circuits. Starting from the exact formula involving a Sommerfeld’s integral, two new approximated analytical formulas, having different levels of approximation, are derived; the advantage of these formulas is that, in such a way, the p.u.l. mutual impedance is expressed only by means of elementary functions so avoiding numerical calculations of the Sommerfeld’s integral. Finally, a comparison between the two proposed formulas is made.

1. INTRODUCTION

It is known that the issue of Extremely Low Frequency electromagnetic interference from power lines/electrified railway lines on pipelines is an important topic related to the electrical safety for staff getting in touch with accessible parts of the pipeline. Risks for safety exist whether the power line/railway line is both in faulty condition or is in normal operating condition; moreover, overvoltage and overcurrent induced on the pipeline may produce damages to the pipeline itself (e.g. insulating coating perforation) or to devices and apparatuses connected to it (e.g. cathodic protection devices). Finally, one has not to forget the AC corrosion risk that is related to the electromagnetic induction generated on the pipe-earth circuit under normal operating condition of the inducing line [1-2].
This paper focuses on the inductive interference generated on long buried metallic structures such as pipelines, ducts, telecommunication cables; the frequencies considered are the typical ones used for transmission and distribution of electrical power i.e. 50-60Hz.

The key parameter for the evaluation of the inductive coupling is the p.u.l. mutual impedance which is directly proportional to the inductive component of the electric field generated by the power line conductors. Such conductors, hung, between two consecutive towers, can be represented, in first approximation, as long horizontal and finite length wires parallel to the soil surface.

In literature, only few contributions, concerning the approach to this problem by means of the finite length wires model, exist; in fact, most of the works are focused on the infinite length wires model whose progenitor is the famous paper by Carson published in 1926 [3].

For a comparison between the two approaches (i.e. finite and infinite length wire models) one can have a look at [4].

It is also worth to mention alternative approaches involving the Finite Element Method [5-8], but they are out of the scope of the present paper.

Among the papers, dealing with mutual coupling between finite length wires with earth return, one has to mention [9-14] where the mutual impedance is expressed by means of the so called “Neumann formulas” involving integrals along the wires path. A different technique is adopted in [15] where the author, by considering an overhead linear horizontal antenna and by applying the “complex image theory”, obtained approximated closed form formulas for electric and magnetic fields valid in the quasi-static range and in the upper semi-space. Nevertheless, no formulas were given for evaluating the field in the lower semi-space i.e. in the soil.

This paper proposes new closed form formulas, built up by means of elementary functions, for evaluating, the inductive component of the electric field in the soil in the ELF range. In particular, thanks to these formulas, one can avoid numerical integration of Sommerfeld’s integral.

2. DERIVATION OF THE FORMULAS

2.1. General

The starting point is evaluating the inductive component of the electric field generated by a finite length overhead wire with earth return carrying a harmonic constant current given by the phasor $I_{pl}$. The overhead wire, assumed having constant height $H$ over the soil, is a representation of a generic power line conductor hung between two consecutive towers (See fig.1).
It is convenient to suppose the wire, having length $L$, as composed by infinitesimal horizontal electric dipoles each one of length $dx$; thus, the field produced by the whole wire can be calculated as sum of all the contributions generated by the single dipoles.

According to [16], the inductive $x$-component of the electric field produced in the soil (medium 2, $z<0$) in the generic point $(x, y, z)$ by a single dipole placed in $(x', y_0, H)$ is proportional to the $x$-component of the electric Hertz potential $\Pi i.e.:

$$
\frac{dE_{2x}^{ind}(x, y, z)}{dx'} = k_2^2 \Pi_{2x}(x, y, z) = \frac{j \omega \mu_0 I dx'}{4\pi} \int_{\lambda_1}^{\lambda_2} \frac{2\lambda e^{-\sqrt{\lambda^2-k_1^2}H} e^{\sqrt{\lambda^2-k_2^2}z}}{\sqrt{\lambda^2-k_1^2} + \sqrt{\lambda^2-k_2^2}} J_0 \left( \sqrt{\lambda^2(x-x')^2 + (y-y_0)^2} \right) d\lambda
$$

(1)

Note that (1) is expressed by means of a Sommerfeld’s integral. In (1), $j$ is the imaginary unit, $J_0$ is the Bessel function of the first kind and order 0 and $k_1$, $k_2$ are respectively given by:

$$
k_1^2 = -j \omega \mu_0 (j \epsilon_0)
$$

(2)

$$
k_2^2 = -j \omega \mu_0 (\sigma_2 + j \omega \epsilon_2)
$$

(3)

In order to obtain the field produced by the wire, one has to integrate expression (1) over the whole wire length, i.e.:

$$
E_{2x}^{ind}(x, y, z) = \int_{L/2}^{L/2} \left[ \frac{j \omega \mu_0 I}{4\pi} \int_0^{\infty} \frac{2\lambda e^{-\sqrt{\lambda^2-k_1^2}H} e^{\sqrt{\lambda^2-k_2^2}z}}{\sqrt{\lambda^2-k_1^2} + \sqrt{\lambda^2-k_2^2}} J_0 \left( \sqrt{\lambda^2(x-x')^2 + (y-y_0)^2} \right) d\lambda \right] dx'
$$

(4)
2.2. Approximated formulas

By introducing suitable hypotheses, it is possible to simplify (1), so that the field \(E_{2x}^{ind}(x, y, z)\) can be expressed by means of an analytical expression much simpler than (4).

The simplifying assumptions can be listed as follows:

- Neglecting displacement currents i.e.: \(k_1 \cong 0\) (in the air) and \(k_2^2 = -j \omega \mu_0 \sigma_2\) (in the earth)

- Due to the extremely low frequencies considered (50-60Hz), to the typical range of values for the soil conductivity \(\sigma_2\) (i.e.: \([10^4, 10^1]\)S/m) and to the typical range of values for the burial depth of pipes (i.e.: \([0.5, 2]\)m), it is possible to write with good approximation:

\[
e^{\sqrt{\lambda^2 - k_2^2 z}} \cong e^{\lambda z}
\]

Therefore, by defining \(\gamma^2 = k_2^2\), formula (1) can be simplified as follows:

\[
dE_{2x}^{ind}(x, y, z) = \frac{j \omega \mu_0 I_d}{4\pi} \int_0^{\infty} \frac{2e^{-\lambda H}}{\lambda + \sqrt{\lambda^2 + \gamma^2}} J_0(\lambda \sqrt{(x-x')^2 + (y-y_0)^2}) d\lambda
\]

In (6), according to [17] a further simplification can be introduced:

\[
\frac{2\lambda}{\lambda + \sqrt{\lambda^2 + \gamma^2}} = 1 - e^{2\gamma} \left[1 + \frac{1}{3} \left(\frac{\lambda}{\gamma}\right)^3 + \cdots\right]
\]

By substituting (7) into (6) and after some algebraic steps, that have been omitted for brevity, one obtains, by the help of certain identities, [18]:

\[
dE_{2x}^{ind}(x, y, z) = \frac{j \omega \mu_0 I_d x'}{4\pi} \left\{ \frac{1}{\sqrt{(x-x')^2 + (y-y_0)^2 + (|z|+H)^2}} + \frac{1}{\sqrt{(x-x')^2 + (y-y_0)^2 + (|z|+H+\frac{2z}{\gamma})^2}} \right\}
\]

In particular, the following identity has been used:

\[
\frac{1}{3\gamma^3} \int_0^{\infty} \lambda^3 e^{-\frac{(|z|+H+\frac{2z}{\gamma})\lambda}{\gamma}} J_0(\lambda \sqrt{(x-x')^2 + (y-y_0)^2}) d\lambda =
\]

\[
\frac{(-1)^3}{3\gamma^3} \frac{d^3}{d(|z|+H+\frac{2z}{\gamma})^3} \left(\frac{1}{\sqrt{(x-x')^2 + (y-y_0)^2 + (|z|+H+\frac{2z}{\gamma})^2}}\right)
\]
In order to obtain the total electric field produced by the overhead wire, one has to integrate expression (8), over whole wire length so obtaining:

\[
E_{2x}^{ind}(x, y, z) = \left\{\begin{array}{l}
\ln \left( \frac{\left(\frac{L}{2}+x+y\right)}{\sqrt{\left(\frac{L}{2}+x+y\right)^2 + \left(y-y_0\right)^2 + \left(|z|+H\right)^2}} \right) + \\
- \ln \left( \frac{\left(\frac{L}{2}+x+y\right)}{\sqrt{\left(\frac{L}{2}+x+y\right)^2 + \left(y-y_0\right)^2 + \left(|z|+H\right)^2}} \right) + \\
\frac{\ln \left( \frac{\left(\frac{L}{2}+x+y\right)}{\sqrt{\left(\frac{L}{2}+x+y\right)^2 + \left(y-y_0\right)^2 + \left(|z|+H\right)^2}} \right)}{\frac{L}{2}+x+y} + \\
\sum_{k=0}^{1} (-1)^k \frac{1}{2k+3} \frac{(x-x')^{2k+3}}{\sqrt{(x-x')^2 + (y-y_0)^2 + \left(|z|+H\right)^2}} + \\
\sum_{k=0}^{2} (-1)^k \frac{2}{2k+1} \frac{(x-x')^{2k+1}}{\sqrt{(x-x')^2 + (y-y_0)^2 + \left(|z|+H\right)^2}} \\
\end{array}\right\}
\]

Also in this case, all the mathematical passages leading from (8) to (10) have been omitted for brevity.

The p.u.l. mutual impedance between the overhead wire with earth return and a buried parallel wire with earth return evaluated in the generic point \((x, y, z)\) belonging to the wire axis is defined by:

\[
Z_m(x, y, z) = \frac{E_{2x}^{ind}(x, y, z)}{I} = \left\{\begin{array}{l}
\ln \left( \frac{\left(\frac{L}{2}+x+y\right)}{\sqrt{\left(\frac{L}{2}+x+y\right)^2 + \left(y-y_0\right)^2 + \left(|z|+H\right)^2}} \right) + \\
+ \ln \left( \frac{\left(\frac{L}{2}+x+y\right)}{\sqrt{\left(\frac{L}{2}+x+y\right)^2 + \left(y-y_0\right)^2 + \left(|z|+H\right)^2}} \right) + \\
\frac{\ln \left( \frac{\left(\frac{L}{2}+x+y\right)}{\sqrt{\left(\frac{L}{2}+x+y\right)^2 + \left(y-y_0\right)^2 + \left(|z|+H\right)^2}} \right)}{\frac{L}{2}+x+y} + \\
\sum_{k=0}^{1} (-1)^k \frac{1}{2k+3} \frac{(x-x')^{2k+3}}{\sqrt{(x-x')^2 + (y-y_0)^2 + \left(|z|+H\right)^2}} + \\
\sum_{k=0}^{2} (-1)^k \frac{2}{2k+1} \frac{(x-x')^{2k+1}}{\sqrt{(x-x')^2 + (y-y_0)^2 + \left(|z|+H\right)^2}} \\
\end{array}\right\}
\]

\[
\left(\begin{array}{c}
x' = \frac{L}{2} \\
x' = -\frac{L}{2}
\end{array}\right)
\]

\[
\left(\begin{array}{c}
x' = \frac{L}{2} \\
x' = -\frac{L}{2}
\end{array}\right)
\]

(11)
Thus, from (11) one gets a function \(Z_m = Z_m(x, y, z)\) that it is composed by four addends; the first and second addends represent the basic contribution (logarithmic terms) while the third and fourth ones may be considered as additional corrective terms; in particular, it can be interesting to estimate their influence on the results.

To this aim, it is convenient to define the p.u.l. mutual impedance \(Z_m' = Z_m'(x, y, z)\) without additional terms, i.e.:

\[
Z_m'(x, y, z) = \frac{j\omega \mu_0 I}{4\pi} \left\{ \ln \left( \frac{L^2 - x + \sqrt{(L^2 - x)^2 + (y - y_0)^2 + (|z| + H)^2}}{-(\frac{L}{2} + x) + \sqrt{\left(\frac{L}{2} + x\right)^2 + (y - y_0)^2 + (|z| + H)^2}} \right) + \right. \\
\left. - \ln \left( \frac{L^2 - x + \sqrt{(L^2 - x)^2 + (y - y_0)^2 + (|z| + H)^2}}{-(\frac{L}{2} + x) + \sqrt{\left(\frac{L}{2} + x\right)^2 + (y - y_0)^2 + (|z| + H)^2}} \right) \right\} (12)
\]

Note that formula (12) is obtained by considering only the first addend, inside square brackets, in formula (7).

### 3. COMPARISON BETWEEN THE MUTUAL IMPEDANCE FORMULAS

In order to calculate the influence of the corrective terms, it is useful to define the following quantities:

\[
\Delta Z_m\% (x, y, z) = \frac{|Z_m(x, y, z)| - |Z_m'(x, y, z)|}{|Z_m(x, y, z)|} \times 100
\]

\[
\Delta \phi (x, y, z) = \arg(Z_m(x, y, z)) - \arg(Z_m'(x, y, z))
\]

Being \(\Delta Z_m\%\) the per cent relative difference between the modulus of \(Z_m\) and \(Z_m'\) while \(\Delta \phi\) is the difference between their arguments.

It is convenient to express the results by means of polar plots being the polar radius \(R = R(x, y)\) given by the formula:

\[
R(x, y) = \sqrt{x^2 + (y - y_0)^2}
\]

The polar plots that follow have been obtained for \(y_0 = 0\), \(H = 20\text{m}\), and \(h_{pipe} = -1.5\text{m}\) (typical burial depth of a pipeline), \(L = 250\text{m}\) (typical distance between two consecutive towers for a High Voltage power line) and the polar angle is counter-clockwise measured starting from the positive \(x\) semi-axis. They have been drawn for different values of soil...
conductivity and in correspondence of different values for the polar radius $R$, i.e.:\(\delta/100, \delta/10, \delta/2, \delta, 2\delta, 3\delta, 5\delta\). Being $\delta$ the skin depth in the earth at the frequency $f = 50\text{Hz}$.

For convenience, the formula for the skin depth in the earth is reported:

$$\delta = \frac{1}{\sqrt{\pi f \mu_2 \sigma_2}} \quad (16)$$

*Figures 2-4* show the polar plots of $\Delta Z_m\%$ and $\Delta \phi$ evaluated for different values of soil conductivity and it can be noticed that:

- The per cent relative difference $\Delta Z_m\%$ ranges approximatively in the interval [-20%, 25%]
- The difference $\Delta \phi$ ranges approximatively in the interval [-30, 130]

These differences occur essentially in the range \([\delta/2, 3\delta]\) for the polar radius while, on the contrary they are negligible outside that range. Therefore, one has:

- if $R<<\delta$ or $R>>\delta$, then the use of $Z_m$ or $Z_m'$ is essentially equivalent
- on the contrary, when $R \sim \delta$, a certain not negligible difference between $Z_m$ and $Z_m'$ can be noticed.

*Fig. 2a. Polar plot of $\Delta Z_m\%$ for different values of polar radius; $\sigma_2=2\cdot10^{-2}\text{S/m}$*
Fig. 2b. Polar plot of $\Delta \phi$ for different values of polar radius; $\sigma_2=2\cdot10^2 S/m$

Fig. 3a. Polar plot of $\Delta Z_{\text{m\%}}$ for different values of polar radius; $\sigma_2=2\cdot10^3 S/m$
Fig. 3b. Polar plot of $\Delta \phi$ for different values of polar radius; $\sigma = 2 \cdot 10^{-3} \text{S/m}$

Fig. 4a. Polar plot of $\Delta Z_{m\%}$ for different values of polar radius; $\sigma = 2 \cdot 10^{-4} \text{S/m}$
4. INFLUENCE OF ADDITIONAL TERMS IN INTERFERENCE CALCULATIONS

A typical and straightforward application of the proposed formulas consists in evaluating the p.u.l. electromotive force (emf) induced on a pipeline-earth circuit under the inductive influence of nearby power lines and then calculating the induced voltage and current on the circuit itself.

Thus, it is interesting to check the influence in using $Z_m'$ instead of $Z_m$ when calculating induced voltage and current on a buried pipeline subjected to the 50Hz inductive coupling from a power line. To this purpose, a simple example consisting of a parallelism between an overhead power line conductor 10km long and a pipeline 5km long is considered. See fig.5 for a simple sketch. The power line conductor-earth circuit carries a constant inducing current $I_{pl}$ while the pipe-earth circuit is closed at its boundaries on the characteristic impedance.
By treating the pipeline-earth circuit as a transmission line circuit, the influence of the electromagnetic field produced by the power line is modelled by a p.u.l. distributed emf generator applied along the pipeline-earth circuit. In the present example, being inducing and induced line parallel, such a generator is related to the above defined p.u.l. mutual impedances respectively by:

\[ e(x) = Z_m(x, D)I_{pl} \]
\[ e'(x) = Z'_m(x, D)I_{pl} \]  \hspace{1cm} (17)

The theory and the formulas for evaluating induced voltage and current on a transmission line circuit under the influence of external electromagnetic fields can be found in [19] and to it one can refer for details.

In order to compare the results, it is worthwhile to define the following quantities:

- \( U(x) \) and \( I(x) \) induced voltage and current obtained when applying the generator \( e(x) \) to the pipeline-earth circuit
- \( U'(x) \) and \( I'(x) \) induced voltage and current obtained when applying the generator \( e'(x) \) to the pipeline-earth circuit
- \( \Delta U\% (x) \) the per cent relative difference relevant to the voltage that is:
  \[ \Delta U\% (x) = \frac{|U(x)| - |U'(x)|}{|U(x)|} \times 100 \]  \hspace{1cm} (18)
- \( \Delta I\% (x) \) the per cent relative difference relevant to the current that is:
  \[ \Delta I\% (x) = \frac{|I(x)| - |I'(x)|}{|I(x)|} \times 100 \]  \hspace{1cm} (19)

In figures 6-8 the quantities \( \Delta U\% (x) \) and \( \Delta I\% (x) \) have been plotted versus pipeline progressive for different values of soil conductivity and in correspondence of different values for the lateral distance \( D \) (see fig. 5) between inducing and induced circuit i.e.: \( \delta 100, \delta 10, \delta 2, \delta 2\delta, 3\delta, 5\delta \).
Fig. 6a. $\Delta U_\%$ versus pipeline progressive for different values of $D$; $\sigma_2 = 2 \cdot 10^{-2} \text{S/m}$

Fig. 6b. $\Delta I_\%$ versus pipeline progressive for different values of $D$; $\sigma_2 = 2 \cdot 10^{-2} \text{S/m}$
Fig. 7a. $\Delta U\%$ versus pipeline progressive for different values of $D$; $\sigma_2=2 \cdot 10^{-3} \text{S/m}$

Fig. 7b. $\Delta I\%$ versus pipeline progressive for different values of $D$; $\sigma_2=2 \cdot 10^{-3} \text{S/m}$
Fig. 8a. $\Delta U_{\%}$ versus pipeline progressive for different values of $D$; $\sigma_2=2\times10^{-4}S/m$

Fig. 8b. $\Delta I_{\%}$ versus pipeline progressive for different values of $D$; $\sigma_2=2\times10^{-4}S/m$
Figures 6-8 show that, in case of parallelism between power line and pipeline, depending on the lateral distance $D$ and on the earth conductivity, the relative per cent difference in evaluating the interference results by using formula (10) or (11) can range approximatively:

- in the interval $[-40\%, 40\%]$ for the induced voltage
- in the interval $[-10\%, 20\%]$ for the induced current

Hence, in certain specific cases, depending on the lateral distance between power line and pipeline and especially with soil having low-medium conductivity, not negligible differences in using $Z_{m}'$ instead of $Z_m$ can be found when calculating induced voltage and current.

Furthermore, it is useful to notice that in [4] it is shown that the results obtained by using $Z_m$ are consistent with the ones coming from the application of the formulas reported in [9] and [11].

The main advantage in using formulas (11) or (12) instead of the ones given in References [9-13] which are based on Neumann integrals, consists in their greater simplicity and in an easier evaluation of the induced emf on the pipeline-earth circuit.

5. CONCLUSIONS

In this paper two new and convenient closed form formulas for evaluating, in the ELF range, the p.u.l. mutual impedance between an overhead finite length wire and a buried wire, both with earth return, are proposed. The two different formulas yield values having per cent relative differences in the range $[-20\%, 25\%]$.

Moreover, when evaluating the interference on the victim circuit (i.e. induced voltage and current), the use of the formula without corrective terms instead of the one with corrective terms may lead, in specific cases, to not negligible differences in the results. Thus, in applications, it is preferable to use the formula (12) instead of (11).

REFERENCES


[4] G. Lucca, *Different approaches in calculating AC inductive interference from power lines on*


